CIVL 7012/8012

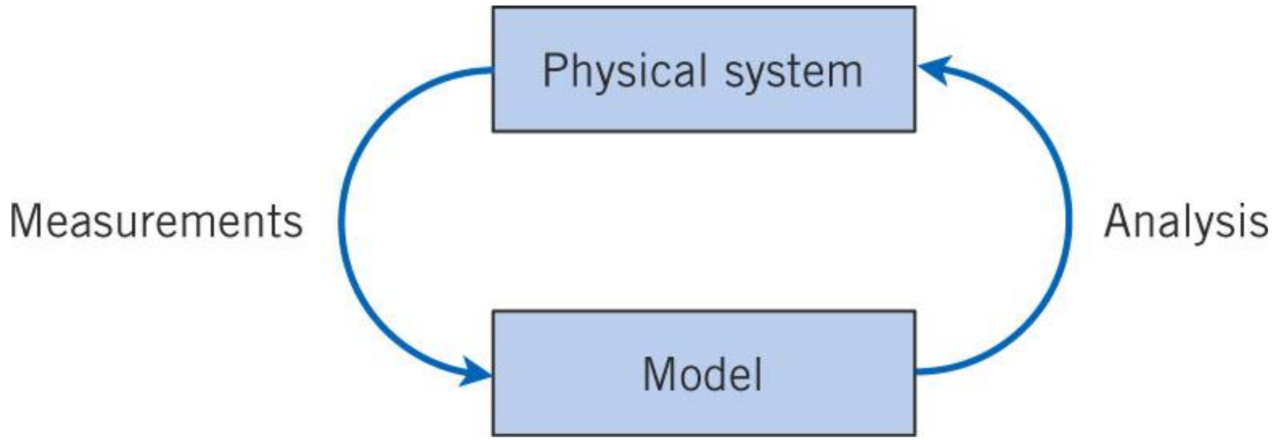
Basic Laws and Axioms of Probability

Why are we studying probability and statistics?

- How can we quantify risks of decisions based on samples from a population?
- How should samples be selected to support good decisions?
- How do we design an experiment so that we obtain the information we need?
- How can we model a problem that has ‘noise’?

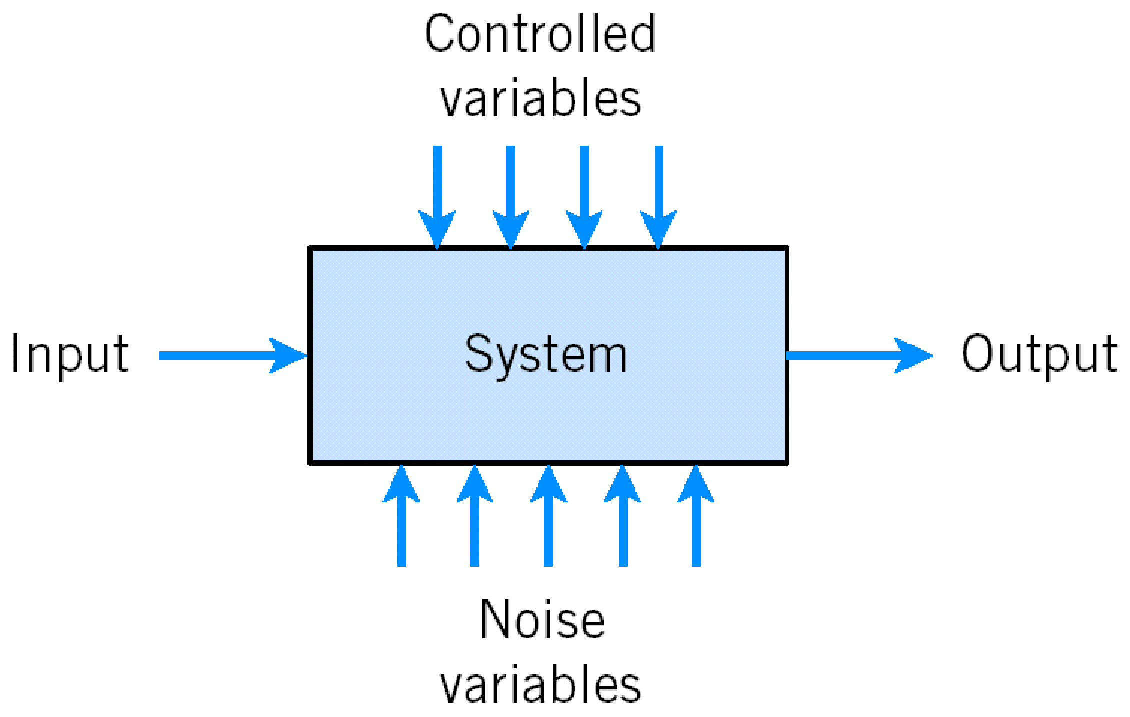
Random Experiment

The goal is to understand, quantify and model the variation affecting a physical system's behavior. The model is used to analyze and predict the physical system's behavior as system inputs affect system outputs. The predictions are verified through experimentation with the physical system.



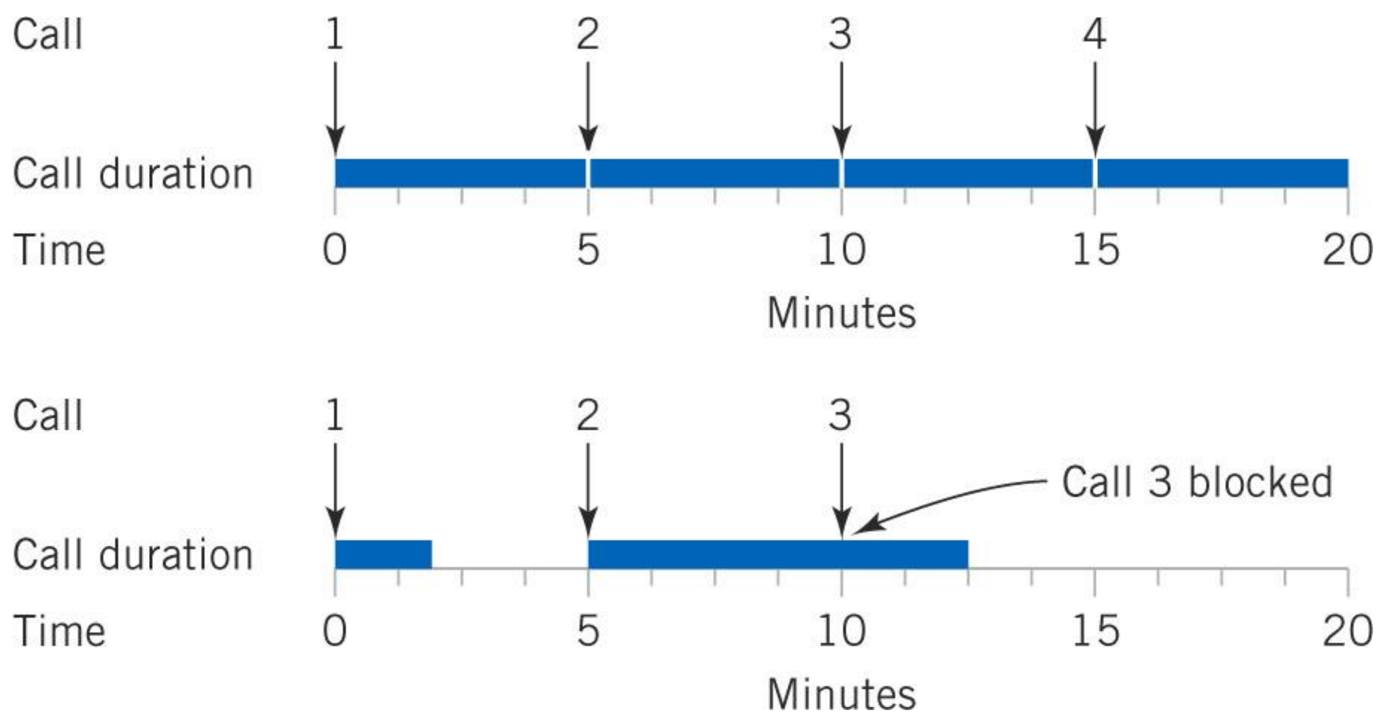
Random Experiment

A *random experiment* can result in different outcomes every time it is repeated, even though the experiment is always repeated in the same manner.



Random Experiment

- Ex. Call center



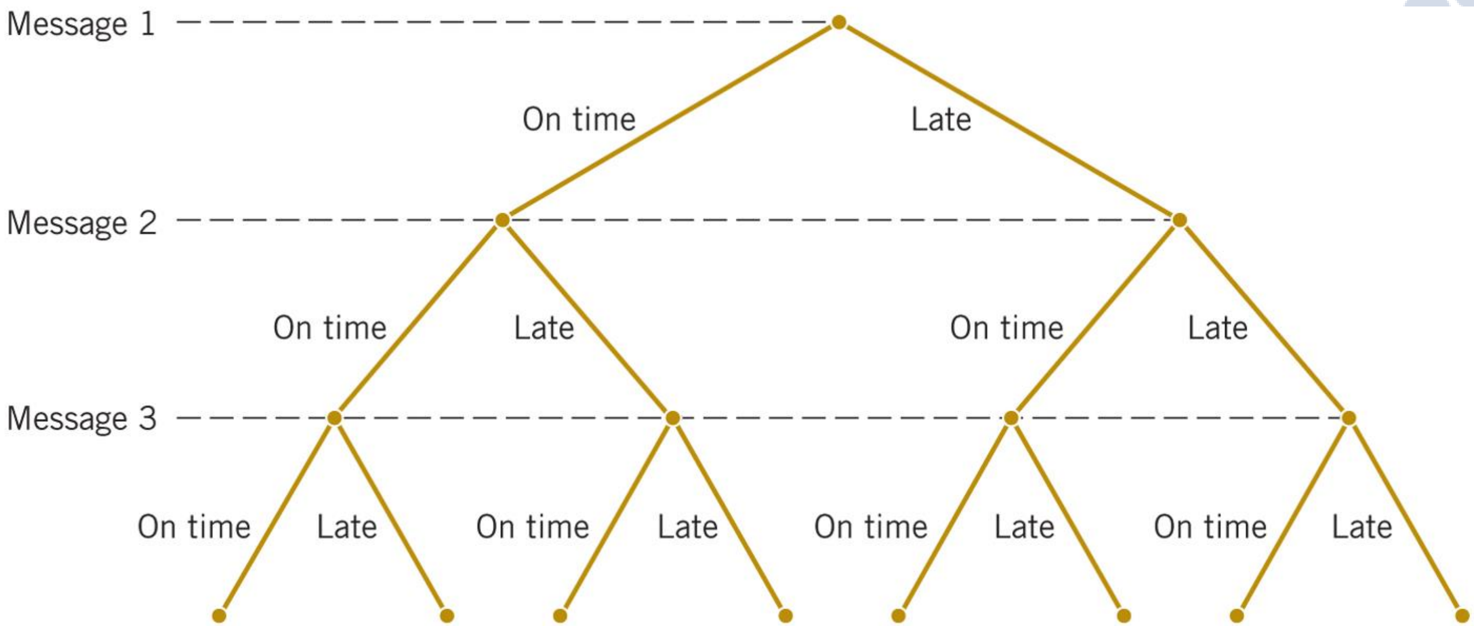
Sample Spaces

- The set of all possible outcomes of a random experiment is called the sample space, *S*.
- *S* is discrete if it consists of a finite or countable infinite set of outcomes.
- *S* is continuous if it contains an interval of real numbers.

Sample Space Defined By A Tree Diagram

Example 2-2: Messages are classified as on-time(o) or late(l). Classify the next 3 messages.

S = {ooo, ool, olo, oll, loo, lol, llo, lll}



Example 2-1: Defining Sample Spaces

- Randomly select a camera and record the recycle time of a flash. $S = R^+ = \{x \mid x > 0\}$, the positive real numbers.
- Suppose it is known that all recycle times are between 1.5 and 5 seconds. Then

  $$S = \{x \mid 1.5 < x < 5\} \text{ is continuous.}$$

- It is known that the recycle time has only three values(low, medium or high). Then $S = \{low, medium, high\}$ is discrete.
- Does the camera conform to minimum recycle time specifications?

  $$S = \{yes, no\} \text{ is discrete.}$$

Basic Laws and Axioms of Probability

DEFINITIONS

- **Experiment** – any action or process that generates observations (e.g. flipping a coin)
- **Trial** – a single instance of an experiment (one flip of the coin)
- **Outcome** – the observation resulting from a trial ("heads")
- **Sample Space** – the set of all possible outcomes of an experiment ("heads" or "tails") (may be discrete or continuous)
- **Event** – a collection of one or more outcomes that share some common trait
- **Mutually Exclusive Events** – events (sets) that have no outcomes in common.
- **Independent Events** – events whose probability of occurrence are unrelated
- **Null Set or Impossible Event** – an empty set in the sample space

Events are Sets of Outcomes

- An event (E) is a subset of the sample space of a random experiment.
- Event combinations
  - The Union of two events consists of all outcomes that are contained in one event <u>or</u> the other, denoted as $E_1 \cup E_2$.
  - The Intersection of two events consists of all outcomes that are contained in one event <u>and</u> the other, denoted as $E_1 \cap E_2$.
  - The Complement of an event is the set of outcomes in the sample space that are <u>not</u> contained in the event, denoted as E'.

Example 2-4 Continuous Events

Measurements of the thickness of a part are modeled with the sample space: $S = R^+$.

Let $E_1 = \{x \mid 10 \le x < 12\}$,

Let $E_2 = \{x \mid 11 < x < 15\}$

- Then $E_1 \cup E_2 = \{x \mid 10 \le x < 15\}$
- Then $E_1 \cap E_2 = \{x \mid 11 < x < 12\}$
- Then $E_1' = \{x \mid 0 < x < 10 \text{ or } x \ge 12\}$
- Then $E_1' \cup E_2 = \{x \mid 12 \le x < 15\}$

Example 2-3 Discrete Events

Suppose that the recycle times of two cameras are recorded. Consider only whether or not the cameras conform to the manufacturing specifications. We abbreviate *yes* and *no* as *y* and *n*. The sample space is $S = \{yy, yn, ny, nn\}$.

Suppose, E_1 denotes an event that at least one camera conforms to specifications, then $E_1 = \{yy, yn, ny\}$

Suppose, E_2 denotes an event that no camera conforms to specifications, then $E_2 = \{nn\}$

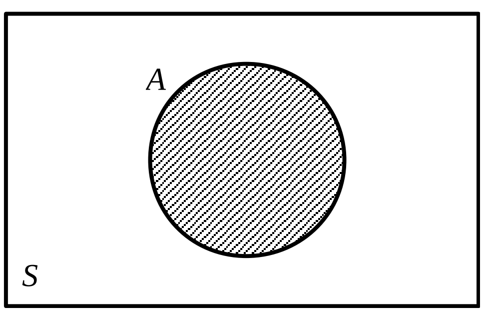
Suppose, E_3 denotes an event that at least one camera does not conform.

then $E_3 = \{yn, ny, nn\}$,

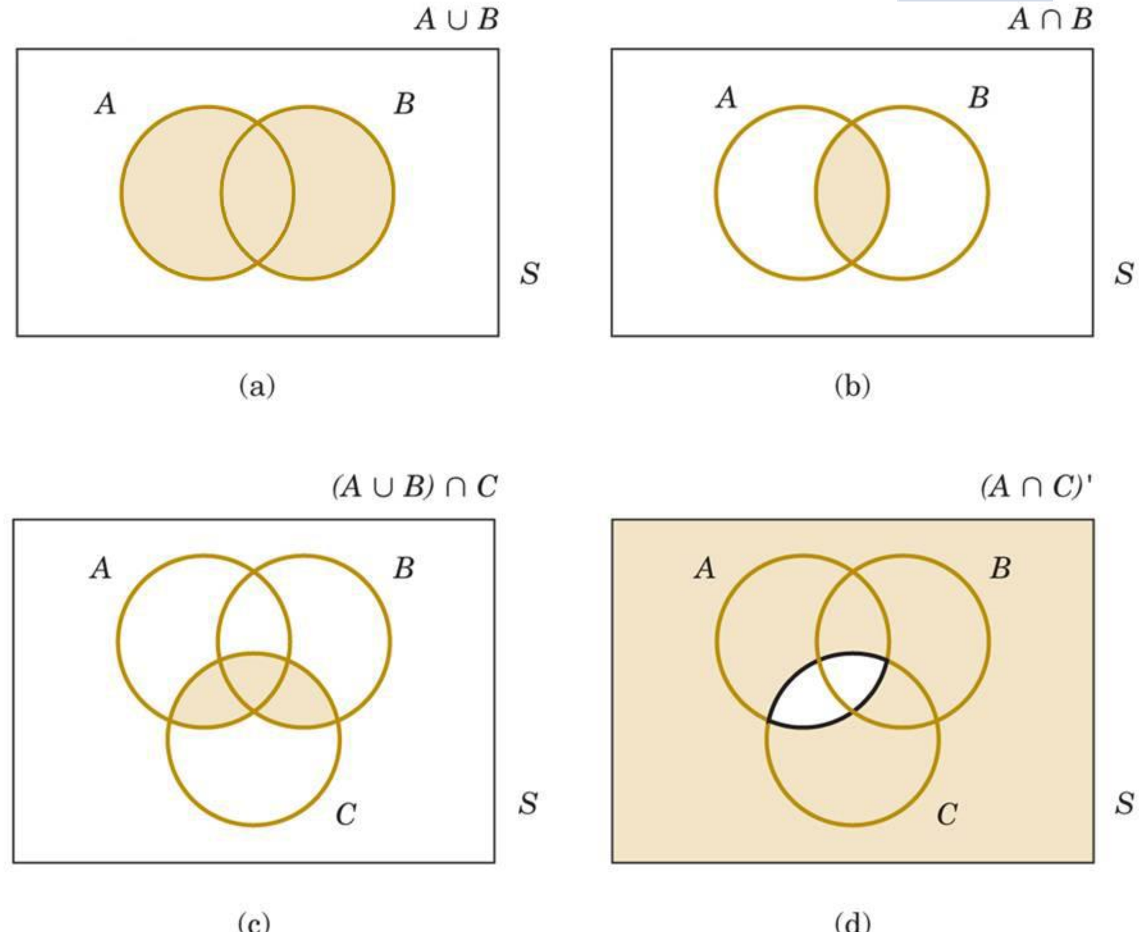
- Then $E_1 \cup E_3 = S$
- Then $E_1 \cap E_3 = \{yn, ny\}$
- Then $E_1' = \{nn\}$

Venn Diagrams

Events *A* & *B* contain their respective outcomes. The shaded regions indicate the event relation of each diagram.



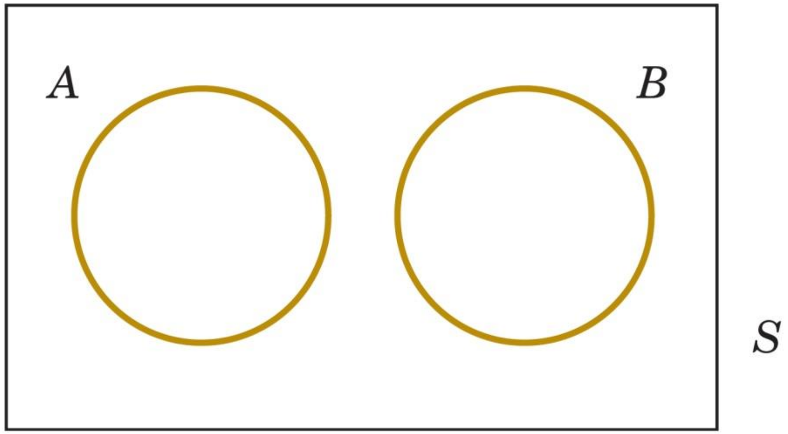
Event A in sample space S.



Sec 2-1.3 Events

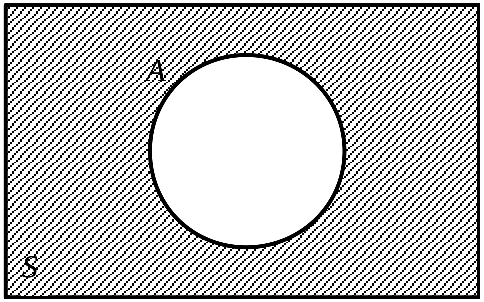
Mutually Exclusive Events

- Events A and B are mutually exclusive because they share no common outcomes.
- The occurrence of one event precludes the occurrence of the other.
- Symbolically, $A \cap B = \varnothing$



Sec 2-1.3

Complement



$$\bar{A} = A'$$

"outcomes in S **not** contained in A"

Mutually Exclusive Events - Laws

- Commutative law (event order is unimportant):
  - $A \cap B = B \cap A$ and $A \cup B = B \cup A$
- Distributive law (like in algebra):
  - $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$
  - $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$
- Associative law (like in algebra):
  - $(A \cup B) \cup C = A \cup (B \cup C)$
  - $(A \cap B) \cap C = A \cap (B \cap C)$

Mutually Exclusive Events - Laws

- DeMorgan's law:
  - $(A \cup B)' = A' \cap B'$ The complement of the union is the intersection of the complements.
  - $(A \cap B)' = A' \cup B'$ The complement of the intersection is the union of the complements.

- Complement law:

$$(A')' = A.$$

Counting Techniques

- There are three special rules, or counting techniques, used to determine the number of outcomes in events.
- They are :
    1. Multiplication rule
    2. Permutation rule
    3. Combination rule
- Each has its special purpose that must be applied properly – the right tool for the right job.

Counting – Multiplication Rule

- Multiplication rule:
  - Let an operation consist of *k* steps and there are
    - n_1 ways of completing step 1,
    - n_2 ways of completing step 2, … and
    - n_k ways of completing step *k*.
  - Then, the total number of ways to perform *k* steps is:
    - $n_1 \cdot n_2 \cdot \ldots \cdot n_k$

Example 2-5 - Web Site Design

- In the design for a website, we can choose to use among:
  - 4 colors,
  - 3 fonts, and
  - 3 positions for an image.

  How many designs are possible?
- Answer via the multiplication rule: $4 \cdot 3 \cdot 3 = 36$

Counting – Permutation Rule

- A permutation is a unique sequence of distinct items.
- If $S = \{a, b, c\}$, then there are 6 permutations
  - Namely: *abc*, *acb*, *bac*, *bca*, *cab*, *cba* (order matters)
- Number of permutations for a set of n items is $n!$
- $n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$
- $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5{,}040$ = FACT(7) in Excel
- By definition: $0! = 1$

Counting–Subset Permutations and an example

- For a sequence of *r* items from a set of *n* items:

$$P_r^n = n(n-1)(n-2)...(n-r+1) = \frac{n!}{(n-r)!}$$

- **Example 2-6**: Printed Circuit Board
- A printed circuit board has eight different locations in which a component can be placed. If four different components are to be placed on the board, how many designs are possible?
- Answer: Order is important, so use the permutation formula with *n* = 8, *r* = 4.

$$P_4^8 = \frac{8!}{(8-4)!} = \frac{8\cdot 7\cdot 6\cdot 5\cdot 4!}{4!} = 8\cdot 7\cdot 6\cdot 5 = 1{,}680$$

Counting - Similar Item Permutations

- Used for counting the sequences when some items are identical.
- The number of permutations of:

  $n = n_1 + n_2 + \ldots + n_r$ items of which

  $n_1, n_2, \ldots, n_r$ are identical.

 is calculated as: $$\frac{n!}{n_1!\,n_2!\,\ldots\,n_r!}$$

Example 2-7: Hospital Schedule

- In a hospital, an operating room needs to schedule three knee surgeries and two hip surgeries in a day. The knee surgery is denoted as *k* and the hip as *h*.
  - How many sequences are there?

    Since there are 2 identical hip surgeries and 3 identical knee surgeries, then

$$\frac{5!}{2!\cdot 3!} = \frac{5\cdot 4\cdot 3!}{2\cdot 1\cdot 3!} = 10$$

  - What is the set of sequences?

    {*kkkhh*, *kkhkh*, *kkhhk*, *khkkh*, *khkhk*, *khhkk*, *hkkkh*, *hkkhk*, *hkhkk*, *hhkkk*}

Counting – Combination Rule

- A combination is a selection of *r* items from a set of *n* where order does not matter.
- If $S = \{a, b, c\}$, $n = 3$, then
  - If $r = 3$, there is 1 combination, namely: *abc*
  - If $r = 2$, there are 3 combinations, namely *ab*, *ac*, and *bc*
- # of permutations ≥ # of combinations
- Since order does not matter with combinations, we are dividing the # of permutations by $r!$, where $r!$ is the # of arrangements of *r* elements.

$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example 2-8: Sampling w/o Replacement-1

- A bin of 50 parts contains 3 defectives and 47 non-defective parts. A sample of 6 parts is selected from the 50 without replacement. How many samples of size 6 contain 2 defective parts?
- First, how many ways are there for selecting 2 parts from the 3 defective parts?

$$C_2^3 = \frac{3!}{2! \cdot 1!} = 3 \text{ different ways}$$

3	=COMBIN(3,2)

Example 2-8: Sampling w/o Replacement-2

- Now, how many ways are there for selecting 4 parts from the 47 non-defective parts?

$$C_4^{47} = \frac{47!}{4! \cdot 43!} = \frac{47 \cdot 46 \cdot 45 \cdot 44 \cdot 43!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 43!} = 178{,}365 \text{ different ways}$$

178,365	=COMBIN(47,4)

Example 2-8: Sampling w/o Replacement-3

- Now, how many ways are there to obtain:
  - 2 from 3 defectives, and
  - 4 from 47 non-defectives?

$$C_2^3 C_4^{47} = 3 \cdot 178{,}365 = 535{,}095 \text{ different ways}$$

535,095	=COMBIN(3,2)*COMBIN(47,4)

Definition of Probability

When conducting an experiment, the probability of obtaining a specific outcome can be defined from its relative frequency of occurrence:

$$P(A) = \lim_{N \to \infty} \left(\frac{n}{N} \right)$$

Example: coin toss

Basic Axioms of Probability

- Let *S* be a sample space. Then *P*(***S***) = 1. $\varnothing$

- For any event *A*, $0 \le P(A) \le 1$.

- If *A* and *B* are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$. More generally, if $A_1, A_2, \ldots\ldots$ are mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \ldots.) = P(A_1) + P(A_2) + \ldots$$

Probability

- Probability is the likelihood or chance that a particular outcome or event from a random experiment will occur.
- Probability is a number in the [0,1] interval.
- A probability of:
  - 1 means certainty
  - 0 means impossibility

Types of Probability

- Subjective probability is a “degree of belief.”

  Example: “There is a 50% chance that I’ll study tonight.”

- Relative frequency probability is based on how often an event occurs over a very large sample space.

  Example: $$\lim_{n \to \infty} \frac{n(A)}{n}$$

Probability Based on Equally-Likely Outcomes

- Whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is $1/N$.
- Example: In a batch of 100 diodes, 1 is laser diode. A diode is randomly selected from the batch. Random means each diode has an equal chance of being selected. The probability of choosing the laser diode is 1/100 or 0.01, because each outcome in the sample space is equally likely.

Probability of an Event

- For a discrete sample space, the *probability of an event E*, denoted by $P(E)$, equals the sum of the probabilities of the outcomes in E.

- The discrete sample space may be:
  - A finite set of outcomes
  - A countably infinite set of outcomes.

Example 2-9: Probabilities of Events

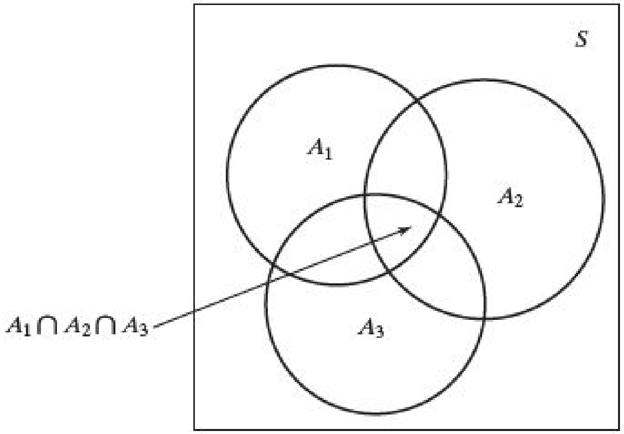
- A random experiment has a sample space $\{a,b,c,d\}$. These outcomes are <u>not</u> equally-likely; their probabilities are: 0.1, 0.3, 0.5, 0.1.
- Let Event $A = \{a,b\}$, $B = \{b,c,d\}$, and $C = \{d\}$
  - $P(A) = 0.1 + 0.3 = 0.4$
  - $P(B) = 0.3 + 0.5 + 0.1 = 0.9$
  - $P(C) = 0.1$
  - $P(A') = 0.6$ and $P(B') = 0.1$ and $P(C') = 0.9$
  - Since event $A \cap B = \{b\}$, then $P(A \cap B) = 0.3$
  - Since event $A \cup B = \{a,b,c,d\}$, then $P(A \cup B) = 1.0$
  - Since event $A \cap C = \{\text{null}\}$, then $P(A \cap C) = 0$

Basic Axioms of Probability

- For any event A, $P(A') = 1 - P(A)$.
- Let $\varnothing$ denote the empty set. Then $P(\varnothing) = 0$.
- If A is an event, and $A = \{O_1, O_2, \ldots, O_n\}$, then $P(A) = P(O_1) + P(O_2) + \ldots + P(O_n)$.
- Addition Rule (for when A and B are not mutually exclusive):

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Basic Axioms of Probability



$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

Addition Rules

- Joint events are generated by applying basic set operations to individual events, specifically:
  - Unions of events, $A \cup B$
  - Intersections of events, $A \cap B$
  - Complements of events, A'
- Probabilities of joint events can often be determined from the probabilities of the individual events that comprise them.

Example 2-10: Semiconductor Wafers

A wafer is randomly selected from a batch that is classified by contamination and location.

- Let H be the event of high concentrations of contaminants. Then $P(H) = 358/940$.
- Let C be the event of the wafer being located at the center of a sputtering tool. Then $P(C) = 626/940$.
- $P(H \cap C) = 112/940$

Contamination	Location of Tool		Total
	Center	Edge	
Low	514	68	582
High	112	246	358
Total	626	314	940

- $P(H \cup C) = P(H) + P(C) - P(H \cap C)$
  $= (358 + 626 - 112)/940$

This is the addition rule.

Probability of a Union

- For any two events *A* and *B*, the probability of union is given by:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- If events *A* and *B* are mutually exclusive, then

$$P(A \cap B) = \varphi,$$

and therefore:

$$P(A \cup B) = P(A) + P(B)$$

Addition Rule: 3 or More Events

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Note the alternating signs.

If a collection of events E_i are pairwise mutually exclusive; that is $E_i \cap E_j = \phi$, for all i, j

Then : $P(E_1 \cup E_2 \cup \ldots \cup E_k) = \sum_{i=1}^{k} P(E_i)$

Conditional Probability

The probability of *A* occurring *given that B has already occurred*:

$$P(A|B)=\frac{P(A\cap B)}{P(B)}$$

The probability of occurrence of the intersection of two sets:

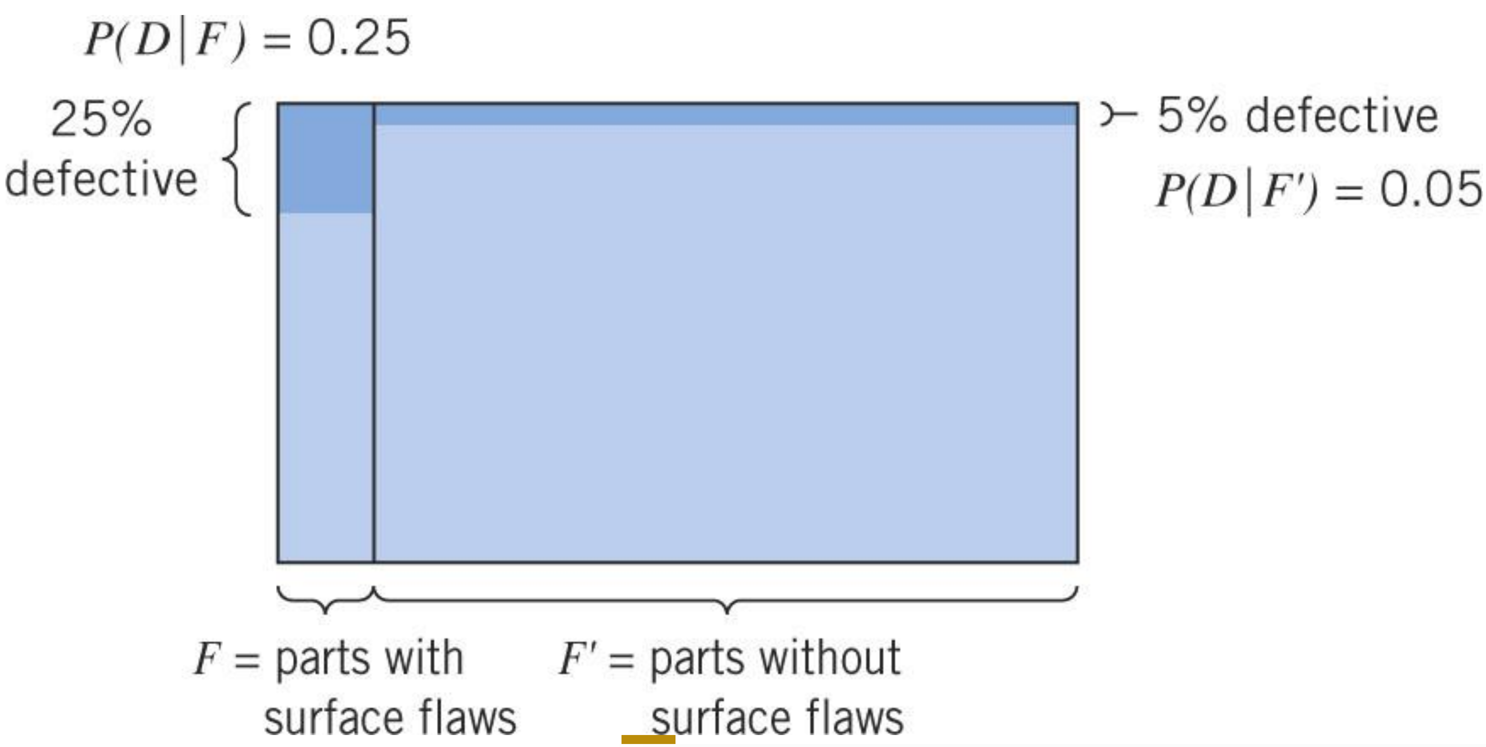
$$P(A\cap B)=P(A|B)\cdot P(B)$$ or P(B|A)P(A)

$$P(A|B)=P(A)$$ **"Independent events"**

If two events are independent, the probability of occurrence of the intersection reduces to:

$$P(A\cap B)=P(A)\cdot P(B)$$ **"The Multiplication Rule"**

Conditional Probability



		Surface Flaws		
		Yes (event F)	No	Total
Defective	Yes (event D)	10	18	28
	No	30	342	372
	Total	40	360	400

Conditional Probability

- $P(B \mid A)$ is the probability of event B occurring, given that event A has already occurred.
- A communications channel has an error rate of 1 per 1000 bits transmitted. Errors are rare, but do tend to occur in bursts. If a bit is in error, the probability that the next bit is also in error is greater than 1/1000.

Conditional Probability Rule

- The conditional probability of an event B given an event A, denoted as $P(B \mid A)$, is:

  $P(B \mid A) = P(A \cap B) \,/\, P(A)$ for $P(A) > 0$.
- From a relative frequency perspective of n equally likely outcomes:
  - $P(A)$ = (number of outcomes in A) / n
  - $P(A \cap B)$ = (number of outcomes in $A \cap B$) / n
  - $P(B \mid A)$ = number of outcomes in $A \cap B$ / number of outcomes in A

Example 2-11

There are 4 probabilities conditioned on flaws in the below table.

		Surface Flaws		
		Yes (event F)	No	Total
Defective	Yes (event D)	10	18	28
	No	30	342	372
	Total	40	360	400

$$P(F) = 40/400 \text{ and } P(D) = 28/400$$

$$P(D \mid F) = P(D \cap F)/P(F) = \frac{10}{400} \Big/ \frac{40}{400} = \frac{10}{40}$$

$$P(D' \mid F) = P(D' \cap F)/P(F) = \frac{30}{400} \Big/ \frac{40}{400} = \frac{30}{40}$$

$$P(D \mid F') = P(D \cap F')/P(F') = \frac{18}{400} \Big/ \frac{360}{400} = \frac{18}{360}$$

$$P(D' \mid F') = P(D' \cap F')/P(F') = \frac{342}{400} \Big/ \frac{360}{400} = \frac{342}{360}$$

Random Samples

- Random means each item is equally likely to be chosen. If more than one item is sampled, random means that every sampling outcome is equally likely.
  - 2 items are taken from $S = \{a,b,c\}$ without replacement.
  - Ordered sample space: $S = \{ab,ac,bc,ba,ca,cb\}$
  - Unordered sample space: $S = \{ab,ac,bc\}$

Example 2-12 : Sampling Without Enumeration

- A batch of 50 parts contains 10 made by Tool 1 and 40 made by Tool 2. If 2 parts are selected randomly*,

a) What is the probability that the 2nd part came from Tool 2, given that the 1st part came from Tool 1?

– $P(E_1) = P(\text{1st part came from Tool 1}) = 10/50$
– $P(E_2 \mid E_1) = P(\text{2nd part came from Tool 2 given that 1st part came from Tool 1})$
$= 40/49$

b) What is the probability that the 1st part came from Tool 1 and the 2nd part came from Tool 2?

– $P(E_1 \cap E_2) = P(\text{1st part came from Tool 1 and 2nd part came from Tool 2})$
$= (10/50) \cdot (40/49) = 8/49$

*Selected randomly implies that at each step of the sample, the items remain in the batch are equally likely to be selected.

Multiplication Rule

- The conditional probability can be rewritten to generalize a multiplication rule.

$$P(A \cap B) = P(B|A) \cdot P(A) = P(A|B) \cdot P(B)$$

- The last expression is obtained by exchanging the roles of *A* and *B*.

Example 2-13: Machining Stages

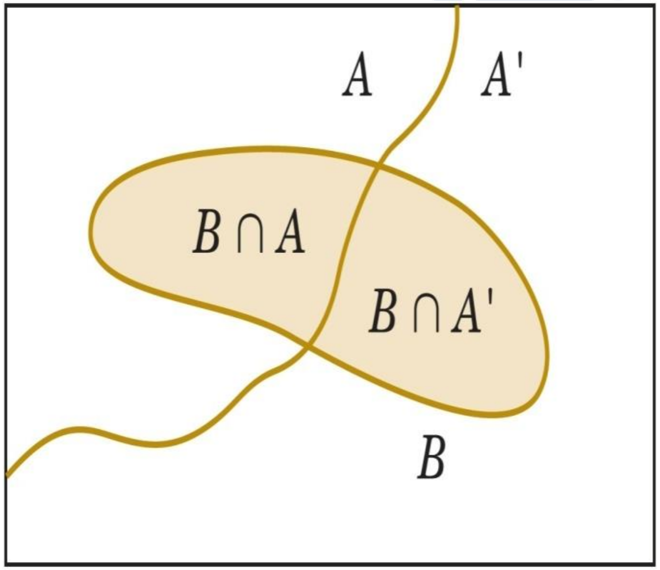
The probability that a part made in the 1^{st} stage of a machining operation meets specifications is 0.90. The probability that it meets specifications in the 2^{nd} stage, given that met specifications in the first stage is 0.95.

What is the probability that both stages meet specifications?

- Let A and B denote the events that the part has met1^{st} and 2^{nd} stage specifications, respectively.
- $P(A \cap B) = P(B \mid A) \cdot P(A) = 0.95 \cdot 0.90 = 0.855$

Two Mutually Exclusive Subsets

- A and A' are mutually exclusive.
- $A \cap B$ and $A' \cap B$ are mutually exclusive
- $B = (A \cap B) \cup (A' \cap B)$



Total Probability Rule

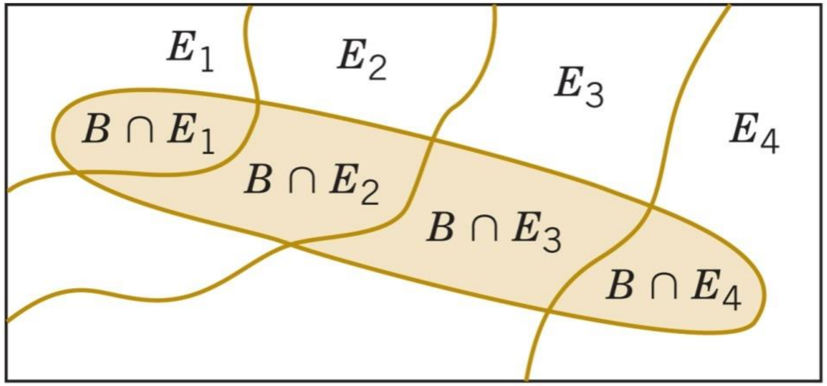
For any two events A and B

$$P(B) = P(B \cap A) + P(B \cap A')$$
$$= P(B \mid A) \cdot P(A) + P(B \mid A') \cdot P(A') \qquad (2\text{-}11)$$

Total Probability Rule (Multiple Events)

- A collection of sets E_1, E_2, … E_k such that
  $E_1 \cup E_2 \cup \ldots\ldots \cup E_k = S$ is said to be **exhaustive**.
- Assume E_1, E_2, … E_k are k mutually exclusive and exhaustive. Then

$$P(B) = P(B \cap E_1) + P(B \cap E_2) + \ldots + P(B \cap E_k)$$
$$= P(B|E_1) \cdot P(E_1) + P(B|E_2) \cdot P(E_2) + \ldots + P(B|E_k) \cdot P(E_k)$$



$$B = (B \cap E_1) \cup (B \cap E_2) \cup (B \cap E_3) \cup (B \cap E_4)$$

Example 2-14: Semiconductor Contamination

Information about product failure based on chip manufacturing process contamination is given below. Find the probability of failure.

Probability of Failure	Level of Contamination	Probability of Level
0.1	High	0.2
0.005	Not High	0.8

Let F denote the event that the product fails.

Let H denote the event that the chip is exposed to high contamination during manufacture. Then

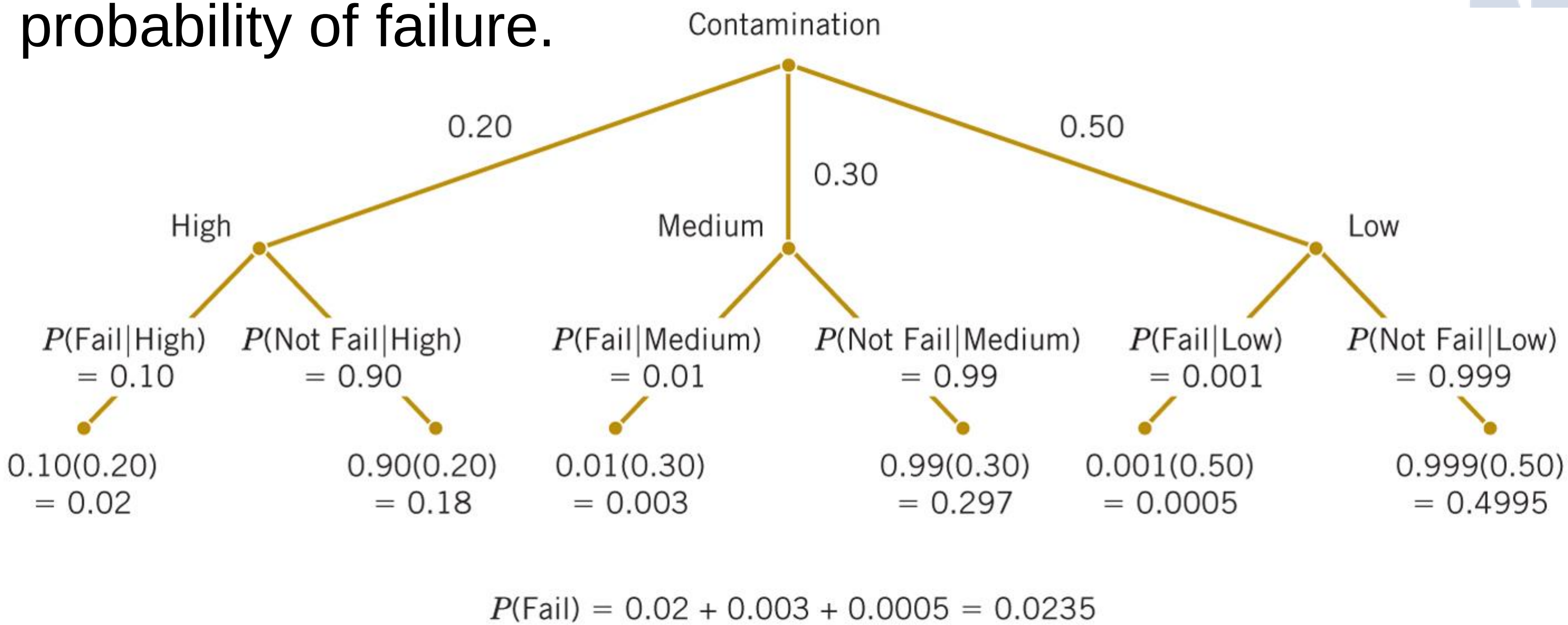
- $P(F \mid H) = 0.100$ and $P(H) = 0.2$, so $P(F \cap H) = 0.02$
- $P(F \mid H') = 0.005$ and $P(H') = 0.8$, so $P(F \cap H') = 0.004$
- $P(F) = P(F \cap H) + P(F \cap H')$ (Using Total Probability rule)

  $= 0.020 + 0.004 = 0.024$

Example 2-15: Semiconductor Failures-1

Probability of Failure	Level of Contamination	Probability of Level
0.100	High	0.2
0.010	Medium	0.3
0.001	Low	0.5

Continuing the discussion of contamination during chip manufacture, find the probability of failure.



Example 2-15: Semiconductor Failures-2

- Let F denote the event that a chip fails
- Let H denote the event that a chip is exposed to high levels of contamination
- Let M denote the event that a chip is exposed to medium levels of contamination
- Let L denote the event that a chip is exposed to low levels of contamination.
- Using Total Probability Rule,

$$
\begin{aligned}
P(F) &= P(F \mid H)P(H) + P(F \mid M)P(M) + P(F \mid L)P(L) \\
&= (0.1)(0.2) + (0.01)(0.3) + (0.001)(0.5) \\
&= 0.0235
\end{aligned}
$$

Event Independence

- Two events are independent if any one of the following equivalent statements is true:
  1. $P(\mathrm{A} \mid \mathrm{B}) = P(A)$
  2. $P(B \mid A) = P(B)$
  3. $P(A \cap B) = P(A) \cdot P(B)$
- This means that occurrence of one event has no impact on the probability of occurrence of the other event.

Example 2-16: Flaws and Functions

Table 1 provides an example of 400 parts classified by surface flaws and as (functionally) defective. Suppose that the situation is different and follows Table 2. Let *F* denote the event that the part has surface flaws. Let *D* denote the event that the part is defective. The data shows whether the events are independent.

TABLE 1 Parts Classified				TABLE 2 Parts Classified (data chg'd)			
	Surface Flaws				Surface Flaws		
Defective	Yes (F)	No (F')	Total	Defective	Yes (F)	No (F')	Total
Yes (D)	10	18	28	Yes (D)	2	18	20
No (D')	30	342	372	No (D')	38	342	380
Total	40	360	400	Total	40	360	400
	$P(D\|F) =$	10/40 =	0.25		$P(D\|F) =$	2/40 =	0.05
	$P(D) =$	28/400 =	0.10		$P(D) =$	20/400 =	0.05
			not same				same
	Events D & F are **dependent**				Events D & F are **independent**		

Independence with Multiple Events

The events E_1, E_2, … , E_k are independent if and only if, for any subset of these events:

$$P(E_{i1} \cap E_{i2} \cap \ldots , \cap E_{ik}) = P(E_{i1}) \cdot P(E_{i2}) \cdot \ldots \cdot P(E_{ik})$$

Example 2-17: Semiconductor Wafers

Assume the probability that a wafer contains a large particle of contamination is 0.01 and that the wafers are independent; that is, the probability that a wafer contains a large particle does not depend on the characteristics of any of the other wafers. If 15 wafers are analyzed, what is the probability that no large particles are found?

Solution:

Let E_i denote the event that the i^{th} wafer contains no large particles, i = 1, 2, …,15.

Then , $P(E_i) = 0.99$.

The required probability is $P(E_1 \cap E_2 \cap \ldots \cap E_{15})$.

From the assumption of independence,

$$
\begin{aligned}
P(E_1 \cap E_2 \cap \ldots \cap E_{15}) &= P(E_1) \cdot P(E_2) \cdot \ldots \cdot P(E_{15}) \\
&= (0.99)^{15} \\
&= 0.86.
\end{aligned}
$$

Bayes' Theorem

- Thomas Bayes (1702-1761) was an English mathematician and Presbyterian minister.
- His idea was that we observe conditional probabilities through prior information.
- Bayes' theorem states that,

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)} \quad \text{for } P(B) > 0$$

Example 2-18

The conditional probability that a high level of contamination was present when a failure occurred is to be determined. The information from Example 2-14 is summarized here.

Probability of Failure	Level of Contamination	Probability of Level
0.1	High	0.2
0.005	Not High	0.8

Solution:

Let *F* denote the event that the product fails, and let *H* denote the event that the chip is exposed to high levels of contamination. The requested probability is *P*(*F*).

$$P(H \mid F) = \frac{P(F \mid H) \cdot P(H)}{P(F)} = \frac{0.10 \cdot 0.20}{0.024} = 0.83$$

$$P(F) = P(F \mid H) \cdot P(H) + P(F \mid H') \cdot P(H')$$
$$= 0.1 \cdot 0.2 + 0.005 \cdot 0.8 = 0.024$$

Bayes Theorem with Total Probability

If E_1, E_2, … E_k are k mutually exclusive and exhaustive events and B is any event,

$$P(E_1 | B) = \frac{P(B|E_1)P(E_1)}{P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \ldots + P(B|E_k)P(E_k)}$$

where $P(B) > 0$

Note : Numerator expression is always one of the terms in the sum of the denominator.

Example 2-19: Bayesian Network

A printer manufacturer obtained the following three types of printer failure probabilities. Hardware $P(H) = 0.3$, software $P(S) = 0.6$, and other $P(O) = 0.1$. Also, $P(F \mid H) = 0.9$, $P(F \mid S) = 0.2$, and $P(F \mid O) = 0.5$.

If a failure occurs, determine if it's most likely due to hardware, software, or other.

$$P(F) = P(F \mid H)P(H) + P(F \mid S)P(S) + P(F \mid O)P(O)$$

$$= 0.9(0.1) + 0.2(0.6) + 0.5(0.3) = 0.36$$

$$P(H \mid F) = \frac{P(F \mid H) \cdot P(H)}{P(F)} = \frac{0.9 \cdot 0.1}{0.36} = 0.250$$

$$P(S \mid F) = \frac{P(F \mid S) \cdot P(S)}{P(F)} = \frac{0.2 \cdot 0.6}{0.36} = 0.333$$

$$P(O \mid F) = \frac{P(F \mid O) \cdot P(O)}{P(F)} = \frac{0.5 \cdot 0.3}{0.36} = 0.417$$

Note that the conditionals given failure add to 1. Because $P(O \mid F)$ is largest, the most likely cause of the problem is in the *other* category.